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A Kinetic Theory Approach to Multiscale Optimisation of Traffic Flow

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Outline

General idea

Control **micro**scopically, optimise **macro**scopically

- How-to:
  1. Implement **vehicle-wise** controls in a particle traffic model
  2. **Upscale** the controlled microscopic dynamics to the macroscopic scale
  3. Deduce a macroscopic model **embedding** vehicle-wise controls
  4. Design control features so as to **optimise** macroscopic traffic trends
- The core of the approach is step 2
  - Methods from the **collisional kinetic theory**

Generic Follow-the-Leader Model

$$\begin{cases} \dot{x}_i = V\left(\dfrac{1}{x_{i+1} - x_i}, \omega_i\right) \\ \dot{\omega}_i = 0 \end{cases} \xrightarrow[\text{headway}]{s_i := x_{i+1} - x_i \geq 0} \begin{cases} \dot{s}_i = V\left(\dfrac{1}{s_{i+1}}, \omega_{i+1}\right) - V\left(\dfrac{1}{s_i}, \omega_i\right) \\ \dot{\omega}_i = 0 \end{cases}$$

- $\omega_i \in \Omega \subseteq \mathbb{R}_+$ Lagrangian marker (trait of single vehicles/drivers)
- $V : \mathbb{R}_+ \times \Omega \to [0, 1]$ dimensionless speed of a vehicle

Binary interactions

$$\begin{cases} s' = s + \gamma \left[V\left(\dfrac{1}{s_*}, \omega_*\right) - V\left(\dfrac{1}{s}, \omega\right) \right] \\ \omega' = \omega \\ s'_* = s_* \\ \omega'_* = \omega_* \end{cases}$$

- $\gamma > 0$ relaxation parameter

Vehicle-Wise Driver-Assist Control Problem

- Add a **driver-assist control** on randomly chosen vehicles:

$$s' = s + \gamma \left[V\left(\frac{1}{s_*}, \omega_*\right) - V\left(\frac{1}{s}, \omega\right) + \boldsymbol{\Theta u} \right] \tag{1}$$

  - $\Theta \sim \mathrm{Bernoulli}(p),\ p \in [0,\ 1]$ **penetration rate**

Binary cost functional

$$J(s', u) := \frac{1}{2}\left(\left| \boldsymbol{H(\rho, w)} - s' \right|^2 + \nu u^2 \right)$$

- $H(\rho, w)$ **recommended headway** (safety distance)
- Optimal feedback control:

$$u^* = \arg\min_u J(s', u) \text{ subject to (1)}$$

Optimally controlled binary interactions

$$s' = s + \frac{\gamma}{\nu + \Theta^2\gamma^2} \left\{ \left[V\left(\frac{1}{s_*}, \omega_*\right) - V\left(\frac{1}{s}, \omega\right) \right] + \Theta^2\gamma \left(H(\rho, w) - s \right) \right\} \tag{2}$$

Enskog-Type Kinetic Description

Kinetic equation

$$\partial_t f + V\left(\frac{1}{s}, \omega\right) \partial_x f = Q_{\mathrm{E}}(f, f)$$

- $f = f(x, s, \omega, t)$ one-particle kinetic distribution function
- **Enskog-type** collisional operator:

$$Q_{\mathrm{E}}(f,f)(x,s,\omega,t) := \frac{1}{2}\int_{\Omega}\int_{\mathbb{R}_+} \left\langle \frac{1}{J} f(x, {}'s, \omega, t) f(x + {}'s, s_*, \omega_*, t) \right\rangle ds_*\, d\omega_*$$
$$- \frac{1}{2}\rho(x+s,t) f(x,s,\omega,t)$$

  - J Jacobian of the transformation from pre- to post-interaction variables
  - $\langle\cdot\rangle$ expectation with respect to the law of Θ
- $\rho(x,t) := \displaystyle\int_{\Omega}\int_{\mathbb{R}_+} f(x,s,\omega,t)\, ds\, d\omega$ **traffic density**

Hydrodynamic Limit

- Hyperbolic scaling: $x \to \frac{x}{\epsilon}$, $t \to \frac{t}{\epsilon}$, $\epsilon > 0$ **"Knudsen" number**
- **Hydrodynamic limit**: $\epsilon \to 0^+$
- Local Maxwellian (under some assumptions on V):

$$M_{\rho,w}(s,\omega) = \rho\delta(s - H(\rho,w)) \otimes \delta(\omega - w)$$

with $w(x,t) := \frac{1}{\rho(x,t)} \int_\Omega \int_{\mathbb{R}_+} \omega f(x,s,\omega,t)\, ds\, d\omega$ **mean Lagrangian marker**

Hydrodynamic equations (GSOM)

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho V \left(\frac{1}{\boldsymbol{H(\rho, w)}}, w \right) \right) = 0 \\ \partial_t (\rho w) + \partial_x \left(\rho w V \left(\frac{1}{\boldsymbol{H(\rho, w)}}, w \right) \right) = 0 \end{cases} \tag{3}$$

- Strictly hyperbolic system if $\partial_\rho H \neq 0$
- Aw-Rascle condition fulfilled if $\partial_\rho H \leq 0$

Hydrodynamic Optimisation

- Maximise the **instantaneous global flux** of vehicles

Small-time-horizon ($\Delta t \ll 1$) cost functional

$$J_{\rho V}(v) = \Delta t \int_{-L}^{L} \left[\rho(x, t+\Delta t) V\left(\frac{1}{\boldsymbol{v(x,t)}}, w(x, t+\Delta t) \right) - \mu F(v(x,t)) \right] dx$$

  - $F = F(v) : \mathbb{R}_+ \to \mathbb{R}_+$ penalisation function, $\mu > 0$ penalisation coefficient
  - $[-L,\, L]$ space domain with periodic boundary conditions

- subject to a discrete-in-time version of (3):

$$\begin{cases} \rho(x, t+\Delta t) = \rho(x,t) - \Delta t \partial_x \left(\rho(x,t) V\left(\frac{1}{\boldsymbol{v(x,t)}}, w(x,t) \right) \right) = 0 \\ w(x, t+\Delta t) = w(x,t) - \Delta t V\left(\frac{1}{\boldsymbol{v(x,t)}}, w(x,t) \right) \partial_x w(x,t) = 0 \end{cases}$$

Analytical Solution to the Hydrodynamic Optimisation Problem

Instantaneous time horizon limit ($\Delta t \to 0^+$)

$$\rho \partial_s V\left(\frac{1}{v^*}, w\right) - \mu F'(v^*) = 0$$

- $H(\rho, w)(x, t) = v^*(x, t) := \arg\max_v J_{\rho V}(v)$ subject to (3)
- For example, with

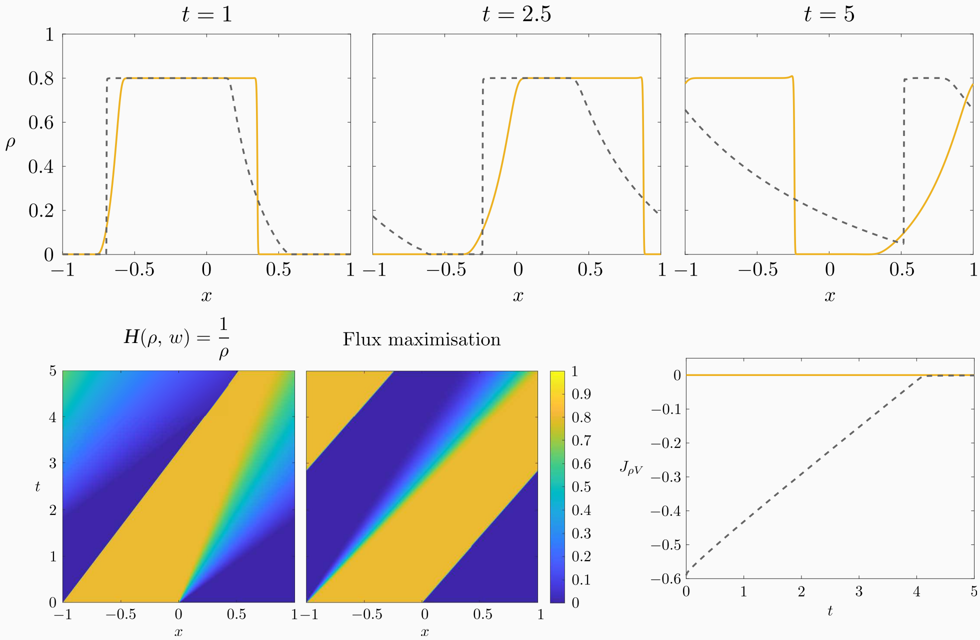
$$V\left(\frac{1}{s}, \omega\right) = \frac{\omega s}{a + s} \quad (a > 0), \qquad F(v) = v\left(\log v - 1\right) + 1$$

it results

$$\left(a + v^*\right)^2 \log v^* = \frac{a}{\mu} \rho w,$$

which admits a unique solution $v^* \geq 1$ for every ρ, $w \geq 0$

Numerical Test



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