A numerical scheme for evolutive Hamilton Jacobi equations on Networks

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joint works with
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Outline

Hamilton Jacobi Equations on Networks

Hamilton Jacobi equation on networks: short review

Stationary case

- Costrained/Relaxation Based [Achdou, Camilli, Cutri, Tchou '14]
- Non symmetric viscosity solutions [Camilli, Schielborn '14]
- Singularly perturbed problem [Achdou, Tchou '15]

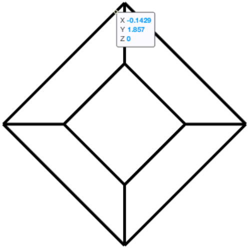
Time dependent

- Flux-limited solutions [Imbert, Monneau '17]
- Kirkoff-based [Lions, Souganidis '17, Morfe '20] (multi-dimensional junction, not require convex Hamiltonian)
- Flux-limited solutions [Siconolfi '22] (without special test functions, and perform tests relative to the equations on different arcs separately)

Numerical method for Hamilton Jacobi equation on networks: short review

- Semi-Lagrangian scheme for eikonal equation [Camilli, Festa, Schieborn '12]
- Finite Difference scheme HJB [Costeseque, Lebacque, Monneau '15]
- Semi-Lagrangian scheme for HJB[C., Festa, Forcadel '20]

Hamilton Jacobi equation on networks



- Arcs: regular simple curves γ parameterized in $[0,1]$
- Network: Γ a subset of $\mathbb{R}^N$ defined as

$$\Gamma = \bigcup_{\gamma \in \mathbf{E}} \gamma([0,1])$$

  where $\mathbf{E}$ if a finite collection of arcs.
- Vertices: $\mathbf{V}$ a subset of $\mathbb{R}^N$ given by initial and terminal points of the arcs, which are the unique points where arcs intersect.
- We fix an orientation $\mathbf{E}^+$ on Γ, and set

$$\mathbf{E}^+_x = \{\gamma \in \mathbf{E}^+ \mid \gamma \text{ incident on } x\}.$$

- Connected network: any two vertices are linked by some arc.
- No loops : arcs with initial and final point coinciding are not admitted.

Assumptions

An Hamiltonian on Γ is a family of Hamiltonians

$$H_\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$$

indexed by arcs such that are

(H1) continuous in both arguments;

(H2) convex in the momentum variable;

(H3) superlinear in the momentum variable, uniformly in s;

Setting of the problem

We consider the family of equations, for any $\gamma \in \mathbf{E}$

$$u_t + H_\gamma(s, u') = 0 \qquad \text{in } (0,1) \times (0,T). \qquad \text{(HJ}\gamma\text{)}$$

with the initial condition

$$u(x,0) = g(x) \qquad \text{for any } x \in \Gamma$$

where $g : \Gamma \to \mathbb{R}$ is a Lipschitz continuous function.

Solution of the problem

In order to uniquely select a continuous function $v : \Gamma \times [0,T) \to \mathbb{R}$, $v \in C(\Gamma \times [0,T))$ solution of (HJγ) for any γ, it has been introduce

$$c_\gamma = -\max_s \min_p H_\gamma(s,p) \qquad \text{for any arc } \gamma,$$

and define

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Definition

A flux limiter is a function $x \mapsto c_x$ from $\mathbf{V}$ to $\mathbb{R}$ satisfying

$$c_x \le \min_{\gamma \in \mathbf{E}_x^+} c_\gamma \qquad \text{for } x \in \mathbf{V}.$$

Reference: Siconolfi '22, and Imbert and Monneau '17

Link between Lagrangian and flux limiter

We define, for each arc $\gamma \in \mathbf{E}_x^+$, the Lagrangian corresponding to H_γ as

$$L_\gamma(s,\alpha) := \max_{p\in\mathbb{R}}(p\alpha - H_\gamma(s,p))$$

Link between Lagrangian and flux limiter

$$c_\gamma = \min_s L_\gamma(s,0)$$

Ref. Pozza and Siconolfi '22, Imbert and Monneau '17

Definition of the problem (HJΓ)

Let $v : \Gamma \times [0, T) \to \mathbb{R}$, $v \in C(\Gamma \times [0, T))$, such that

- $v \circ \gamma$ is a viscosity solution to (HJ$_\gamma$) in $(0, 1) \times (0, T)$, for any γ,
- $v \circ \gamma$ verifies the initial condition: $v(\gamma(s), 0) = g(\gamma(s))$,
- at any $x \in \mathbf{V}$, $t_0 \in (0, T)$:

Definition (Sub-solution at a vertex)

For any $\psi(t) \in C^1(U)$, U neighbourhood of t_0, s.t. $\psi(t_0) = v(x, t_0)$ and $\psi(t) \geq v(x, t)$ for any $t \in U$, ($\psi(t)$ is supertangents to $v(x, \cdot)$ at t_0) satisfy

$$\frac{d}{dt}\psi(t_0) \leq c_x.$$

Reference: Siconolfi '22

Super-solution at a vertex

A at any $x \in \mathbf{V}$, $t_0 \in (0,T)$:

Definition (Super-solution at a vertex)

If exists a C^1 subtangent $\phi(t)$ to $v(x,\cdot)$ at t_0 such that

$$\frac{d}{dt}\phi(t_0) < c_x,$$

then there is an arc γ s.t. $\gamma(1) = x$ and such that all the C^1 subtangents φ in $(1,t_0)$, constrained* to $[0,1]\times[0,T]$, to $v \circ \gamma$ at $(1,t_0)$ satisfy

$$\varphi_t(1,t_0) + H_\gamma(1,\varphi'(1,t_0)) \geq 0.$$

* φ is a constrained supertangent to $[0,1]\times[0,T]$ on (s_0,t_0) if $\varphi(s_0,t_0) = v(\gamma(s_0),t_0)$ and $\varphi(s,t) \geq v(\gamma(s),t)$ in a neighborhood of (s_0,t_0) intersected with $[0,1]\times[0,T]$

Note that the arc γ, with $\gamma(1) = x$ may changes in function of the time.

Well posedness

Let (**H1**)-(**H3**) hold true.

Theorem (A.Siconolfi '22)

Let u, v be continuous sub and supersolution to (HJΓ) respectively, in $\Gamma \times (0, T)$ with $u(\cdot, 0) \le v(\cdot, 0)$ in Γ, then $u \le v$ in $\Gamma \times [0, T)$.

Theorem (A.Siconolfi '22)

For any continuous initial datum g and flux limiter c_x, there exists one and only one continuous solution to (HJΓ) in $(0, T)$.
If g is Lipschitz continuous, the solution is Lipschitz continuous as well.

A numerical scheme for HJ on Networks

An algorithm–preliminary steps

- Given $\Delta x > 0$, $\Delta t > 0$, for $\gamma \in \mathbf{E}^+$ we fix positive integers

$$N_\gamma^\Delta = \left\lfloor \frac{|\gamma(1) - \gamma(0)|}{\Delta x} \right\rfloor > 0 \quad \text{for any } \gamma \in \mathbf{E}^+, \text{ and} \quad N_T^\Delta = \left\lfloor \frac{T}{\Delta t} \right\rfloor > 0$$

- We consider a uniform grid on $[0,1] \times [0,T]$ for each γ, and we set

$$\begin{aligned}
\mathcal{S}_{\Delta,\gamma} &= \{s_i^\gamma = \frac{i}{N_\gamma^\Delta} \mid i = 0, \dots, N_\gamma^\Delta\} \\
\mathcal{T}_\Delta &= \{t_n = \frac{nT}{N_T^\Delta} \mid n = 0, \dots, N_T^\Delta\} \\
\Gamma_\Delta &= \bigcup_{\gamma \in \mathbf{E}^+} \gamma(\mathcal{S}_{\Delta,\gamma}) \times \mathcal{T}_\Delta
\end{aligned}$$

An algorithm–step 0

- We solve numerically the equation (HJγ) in $(0,1) \times (0,T)$ wth initial condition at $t = 0$ given by

$$(g(\gamma(s_0^\gamma)), \cdots, g(\gamma(s_{N_\gamma}^\gamma))) \text{ for any } \gamma \in \mathbf{E}^+$$

and denote by

$$u_\gamma^1(s_i^\gamma) \qquad i = 1, \cdots, N_\gamma$$

the approximate solutions so obtained.

An algorithm–step 0

- We solve numerically the equation (HJγ) in $(0,1) \times (0,T)$ wth initial condition at $t = 0$ given by

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and denote by

$$u_\gamma^1(s_i^\gamma) \qquad i = 1, \cdots, N_\gamma$$

the approximate solutions so obtained.

- We get, for any vertex x, a finite family of values

$$u_\gamma^1(\gamma^{-1}(x)) \qquad \text{for } \gamma \in \mathbf{E}_x^+.$$

An algorithm–step 1

- The compatibility condition between arcs of Γ_x^+ is given by

$$\begin{aligned} a &= \min\{u^1_\gamma(\gamma^{-1}(x)) \mid \gamma \in \mathbf{E}_x^+\} \\ u^1(x) &= \min\{g(x) + c_x\,\Delta t,\, a\}. \end{aligned}$$

An algorithm–step 1

- The compatibility condition between arcs of Γ_x^+ is given by

$$\begin{aligned} a &= \min\{u_\gamma^1(\gamma^{-1}(x)) \mid \gamma \in \mathbf{E}_x^+\} \\ u^1(x) &= \min\{g(x) + c_x\,\Delta t,\, a\}. \end{aligned}$$

- We have therefore determined, for any arc $\gamma \in \mathbf{E}^+$, a vector

$$u_\gamma^1 = (u^1(0), u_\gamma^1(s_1^\gamma), \cdots, u_\gamma^1(s_{N_\gamma-1}^\gamma), u^1(1))$$

to use as initial value in the next step.

An algorithm– step $n < N_T$

- Given u^{n-1}, we solve numerically the equation (HJγ) in any $\gamma \in \mathbf{E}^+$ for one time step, and we get

$$u_\gamma^n = (u_\gamma^n(s_0^\gamma), u_\gamma^n(s_1^\gamma), \cdots, u_\gamma^n(s_{N_\gamma -1}^\gamma), u_\gamma^n(s_{N_\gamma}^\gamma))$$

An algorithm– step $n < N_T$

- Given u^{n-1}, we solve numerically the equation (HJγ) in any $\gamma \in \mathbf{E}^+$ for one time step, and we get

$$u^n_\gamma = (u^n_\gamma(s^\gamma_0), u^n_\gamma(s^\gamma_1), \cdots, u^n_\gamma(s^\gamma_{N_\gamma - 1}), u^n_\gamma(s^\gamma_{N_\gamma}))$$

- We compute the value at any vertex x setting

$$\begin{aligned} a &= \min\{u^n_\gamma(\gamma^{-1}(x)) \mid \gamma \in \mathbf{E}^+_x\} \\ u^n(x) &= \min\{u^n(x) + c_x \, \Delta t, \, a\}, \end{aligned}$$

- We iterate untill $n = N_T$

A SL numerical scheme

On each arc $\gamma \in \mathbf{E}^+$, the DPP principle holds

$$v_\gamma(s, t_{n+1}) = \inf_{\mu \in L^\infty} \left\{ v_\gamma(y_s(\Delta t), t_n) + \int_{t_n}^{t_{n+1}} L_\gamma(y_s(\tau), \mu(\tau)) d\tau \right\}.$$

where $y_s(\tau)$ solves

$$\dot{y}(\tau) = -\mu(\tau)\ \tau \in (t_n, t_{n+1}), \text{ for a.e.} \quad y(t_{n+1}) = s$$

Inside each arc γ, we discretize the backward trajectory as

$$y_s(\Delta t) \simeq s - \Delta t \mu(t_{n+1}) = s - \Delta t \alpha$$

and we discretize DPP to solve (HJγ) by defining on each arc $\gamma \in \mathbf{E}^+$

$$S_{\Delta,\gamma}[u](s, t_n) = \min_{\frac{s-1}{\Delta t} \leq \alpha \leq \frac{s}{\Delta t}} \{ u(\pi_{\Delta,\gamma}(s - \Delta t \alpha), t_n) + \Delta t L_\gamma(s, \alpha) \} \qquad (1)$$

where $\pi_{\Delta,\gamma}$ is a constant or linear interpolation on the space grid of the discretize backward trajectory

Ref. Falcone, Ferretti 2014

A SL numerical scheme

We define the numerical operator: if $x \in \Gamma_\Delta \setminus \mathbf{V}$

$$S_\Delta[u](x,t) = \{S_{\Delta,\gamma}[u \circ \gamma](\gamma^{-1}(x),t) \mid \gamma \in \mathbf{E}_x^+\},$$

if instead $x \in \mathbf{V}$, a vertex,

$$\begin{array}{rcl} \widetilde{S}_\Delta[u](x,t) & = & \min\{S_{\Delta,\gamma}[u \circ \gamma](\gamma^{-1}(x),t) | \, \gamma \in \mathbf{E}_x^+\} \\ S_\Delta[u](x,t) & = & \min\{\widetilde{S}_\Delta[u](x,t), u(x,t) + c_x \Delta t\} \end{array}$$

We finally consider the following evolutive explicit scheme corresponding to the above discretization of (HJΓ):

$$\left\{ \begin{array}{rcl} u(x,0) & = & g(x) \\ u(x,t) & = & S_\Delta[u](x,t-\Delta t)) \end{array} \right. \qquad (\text{HJ}\Gamma_\Delta)$$

for $(x,t) \in \Gamma_\Delta \cap \Gamma \times (0,T]$. Let call u_Δ the solution of (HJΓ_Δ)

Property of the numerical operators

Proposition

Let $\Delta = (\Delta x, \Delta t) \to (0,0)$ *with* $\Delta x / \Delta t \to 0$*, then for any arc* γ *and for any function* $\psi : [0,1] \times [0,T] \to \mathbb{R}$ *of class* C^1 *we have*

$$\frac{\psi(s,t) - S_{\Delta,\gamma}[\psi](s,,t-\Delta t)}{\Delta t} \to \psi_t(s,t) + H_\gamma(s,\psi'(s)) \quad \text{as} \quad \Delta \to 0$$

locally uniformly in $(0,1) \times (0,T]$.

Proposition

S_Δ *is monotone and invariant by addition of constants*

i) *given* $\Delta = (\Delta x, \Delta t)$*, and* u_1, $u_2 \in B(\Gamma_\Delta)$ *with* $u_1 \le u_2$*, we have*

$$S_\Delta[u_1](x,t) \le S_\Delta[u_2](x,t) \qquad \text{for all } (x,t) \in \Gamma_\Delta;$$

ii) *given* Δ *and* $u \in B(\Gamma_\Delta)$*, we have for any constant* C*, and* $(x,t) \in \Gamma_\Delta$.

$$S_\Delta[u+C](x,t) = S_\Delta[u](x,t) + C$$

Convergence Analysis

We further assume

(H1) continuous in both arguments;

(H2) convex in the momentum variable;

(H3) superlinear in the momentum variable, uniformly in s;

(H4) $s \mapsto H_\gamma(s,\mu)$ is Lipschitz continuous

Theorem (Sub-solution property)

Let $\Delta = (\Delta x, \Delta t) \to (0,0)$ *with* $\Delta x/\Delta t \to 0$, *then*

$$u_\Delta \to v$$

locally uniformly in $\Gamma \times [0,T)$, v *is Lipschits and it is viscosity sub-solution to (HJ*Γ*) with initial datum* g *.*

The difficult point is to show the supersolution condition at the vertices.

Supersolution condition at the vertices

Let $x = \gamma(0)$ be a vertex s.t. $\psi'(t_0) < c_x$ for some C^1 subtangent ψ to $v(x, \cdot)$ at $t_0 \in (0, T]$, and $t_m \in \mathcal{T}_{\Delta_m}$ with t_m converging t_0, let's call $u_m := u_{\Delta_m}$ and $\tilde{v} := \lim_{m \to \infty} u_m$then

- there is an arc $\gamma \in \mathbf{E}_x$ such that

$$u_m(x, t_m) = S_{\Delta, \gamma}[u_m \circ \gamma](\gamma^{-1}(x), t_m - \Delta_m t) \tag{2}$$

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- then we can define an optimal discrete trajectories $\xi_m(s)$, backward in time, which stays in the arc γ for a time $\delta > 0$

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- then we can define an optimal discrete trajectories $\xi_m(s)$, backward in time, which stays in the arc γ for a time $\delta > 0$
- $\xi_m(s)$ are uniformly convergent to a trajectory ξ and, since $\tilde{v}$ is subsolution and L_γ lower semiconituous, verify

$$\int_{t_0-\delta}^{t_0} L_\gamma(\xi, \dot{\xi})\, dt = \tilde{v} \circ \gamma(0, t_0) - \tilde{v} \circ \gamma(\xi(t_0 - \delta), t_0 - \delta).$$

(see Pozza, Siconolfi '22)

Supersolution condition at the vertices

Let $x = \gamma(0)$ be a vertex s.t. $\psi'(t_0) < c_x$ for some C^1 subtangent ψ to $v(x, \cdot)$ at $t_0 \in (0, T]$, and $t_m \in \mathcal{T}_{\Delta_m}$ with t_m converging t_0, let's call $u_m := u_{\Delta_m}$ and $\tilde{v} := \lim_{m\to\infty} u_m$then

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(see Pozza, Siconolfi '22)

- $\xi(t_0 - \delta) \neq 0$(no oscillations for the definition of c_x)
  $\xi(t_0 - \delta) \neq 1$ (because δ can be chosen small enough)

Supersolution condition at the vertices

- if by contradiction that there is a C^1 subtangent φ, to $u \circ \gamma$ at $(0, t_0)$ with
$$\varphi_t(0, t_0) + H_\gamma(0, \varphi'(0, t_0)) < 0,$$
by Perron-Ishii method, there exist a new subsoution w s.t.

$$\begin{aligned} w(0,t_0) - w(\xi(t_0-\delta), t_0-\delta) &> \tilde{v} \circ \gamma(0, t_0) - \tilde{v} \circ \gamma(\xi(t_0-\delta), t_0-\delta). \\ &= \int_{t_0-\delta}^{t_0} L_\gamma(\xi, \dot{\xi})\, dt \end{aligned}$$

Supersolution condition at the vertices

- if by contradiction that there is a C^1 subtangent φ, to $u \circ \gamma$ at $(0, t_0)$ with

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$$\begin{aligned} w(0, t_0) - w(\xi(t_0 - \delta), t_0 - \delta) & > \tilde{v} \circ \gamma(0, t_0) - \tilde{v} \circ \gamma(\xi(t_0 - \delta), t_0 - \delta). \\ & = \int_{t_0-\delta}^{t_0} L_\gamma(\xi, \dot{\xi})\, dt \end{aligned}$$

- this is a contradiction, since w be a subsolution , implyes

$$w(s_2, t_2) - w(s_1, t_1) \le \int_{t_1}^{t_2} L_\gamma(\eta, \dot{\eta})\, dt$$

for any (s_i, t_i), $i = 1, 2$, with $t_1 < t_2$, any curve $\eta : [t_1, t_2] \to [0, 1]$ joining s_1 to s_2

Main result

Theorem

Let $\Delta = (\Delta x, \Delta t) \to (0,0)$ *with* $\Delta x/\Delta t \to 0$, *then*

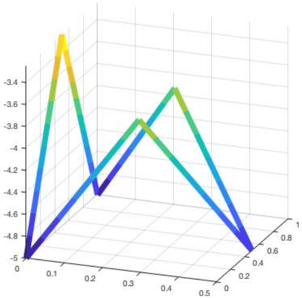
$$u_\Delta \to v$$

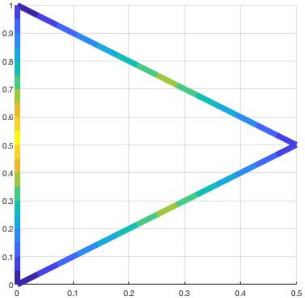
locally uniformly in $\Gamma \times [0,T)$, v *viscosity solution to (HJ*Γ*) with Lipscht continuous initial datum* g .

Numerical tests

Test 1: very simple network

We consider a triangle as network, $L_{\gamma_i}(x,q) = \frac{q^2}{2}$, for all $i = 1, 2, 3$, admissible flux limiters $c_1 = c_2 = c_3 = -5$ and as initial condition $g = 0$





Approximated solution at final time $T = 1$with $c_1 = c_2 = c_3 = -5$, with $\Delta x = 0.05$ and $\Delta t = \frac{\Delta x}{2}$

The hyperbolic CFL condition $\max_{\gamma,s} |u'(\gamma(s))|\Delta t \leq \Delta x$ is not verified, since the Courant number $\nu = \max_{\gamma,s} |u'(\gamma(s))| \frac{\Delta t}{\Delta x} = \sqrt{10}/2 > 1$.

Comparison with pure SL scheme

Comparison with pure SL scheme (C., Festa, Forcadel)

Δx	E^∞	E^1	time	E^∞	E^1	time
$1.00 \cdot 10^{-1}$	$3.57 \cdot 10^{-2}$	$1.38 \cdot 10^{-2}$	$0.01s$	$2.10 \cdot 10^{-5}$	$5.18 \cdot 10^{-6}$	$0.08s$
$5.00 \cdot 10^{-2}$	$1.74 \cdot 10^{-2}$	$6.60 \cdot 10^{-3}$	$0.07s$	$1.19 \cdot 10^{-5}$	$1.02 \cdot 10^{-6}$	$0.41s$
$2.50 \cdot 10^{-2}$	$8.56 \cdot 10^{-3}$	$3.25 \cdot 10^{-3}$	$0.47s$	$9.79 \cdot 10^{-6}$	$2.57 \cdot 10^{-7}$	$2.10s$
$1.25 \cdot 10^{-2}$	$4.25 \cdot 10^{-3}$	$1.61 \cdot 10^{-3}$	$3.54s$	$4.29 \cdot 10^{-7}$	$1.15 \cdot 10^{-7}$	$14.0s$
$6.25 \cdot 10^{-3}$	$2.11 \cdot 10^{-3}$	$8.15 \cdot 10^{-4}$	$28.3s$	$3.49 \cdot 10^{-8}$	$8.45 \cdot 10^{-9}$	$99.0s$

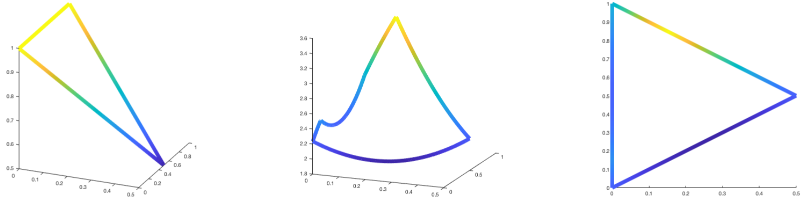
Table: Columns 2-4 shows errors, and computational time for the new scheme. Columns 5-7 shows errors and computational time for the SL scheme

Remark: In the numerical simulation, we have used a linear interpolation. This led to a truncation errors: $\frac{\Delta x^2}{\Delta x} + \Delta t$, which means that for $\Delta t = O(\Delta x)$ a first order rate of convergence is expected

Test 1: very simple network

Let us now choose cost functions depending on x, as

$$L(x,q) = \begin{cases} \frac{|q|^2}{2} + 5|x_1 - 0.5|^2 + 5|x_2 - 0.5|^2 + 10x_2^2 & \text{if } x \in \gamma_2, \\ \frac{|q|^2}{2} + 5|x_1 - 0.5|^2 + 5|x_2 - 0.5|^2 + 10x_2^2 & \text{if } x \in \gamma_3, \\ \frac{|q|^2}{2} + 5|x_1 - 0.5|^2 + 5|x_2 - 0.5|^2 & \text{if } x \in \gamma_1. \end{cases}$$



Initial condition (left) and approximated solution (center, right) at final time $T = 1$ with $c_1 = c_2 = c_3 = 2$, computed with $\Delta x = 6.25 \cdot 10^{-2}$ and $\Delta t = \frac{\Delta x}{2}$.

Test 1: very simple network

Δx	E^∞	E^1	time	E^∞	E^1	time
$1.00 \cdot 10^{-1}$	$1.93 \cdot 10^{-1}$	$1.49 \cdot 10^{-1}$	$0.03s$	$1.93 \cdot 10^{-1}$	$1.34 \cdot 10^{-1}$	$0.28s$
$5.00 \cdot 10^{-2}$	$1.07 \cdot 10^{-1}$	$7.57 \cdot 10^{-2}$	$0.16s$	$1.04 \cdot 10^{-1}$	$6.94 \cdot 10^{-2}$	$1.19s$
$2.50 \cdot 10^{-2}$	$5.77 \cdot 10^{-2}$	$7.67 \cdot 10^{-2}$	$0.70s$	$5.34 \cdot 10^{-2}$	$3.43 \cdot 10^{-2}$	$7.66s$
$1.25 \cdot 10^{-2}$	$2.90 \cdot 10^{-2}$	$1.73 \cdot 10^{-2}$	$5.26s$	$2.55 \cdot 10^{-2}$	$1.69 \cdot 10^{-2}$	$56.3s$
$6.25 \cdot 10^{-3}$	$1.42 \cdot 10^{-2}$	$7.85 \cdot 10^{-3}$	$40.1s$	$1.17 \cdot 10^{-2}$	$7.46 \cdot 10^{-3}$	$444s$

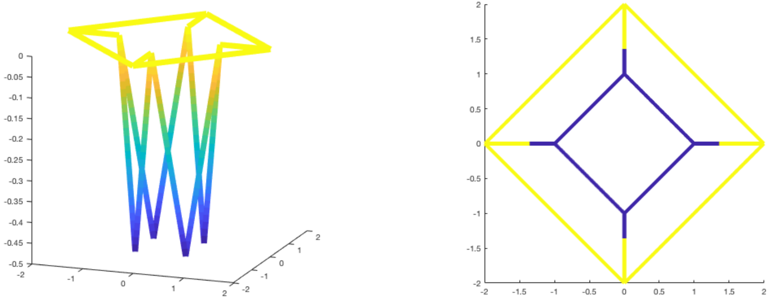
L^∞ and L^1 errors computed $\Delta t = \Delta x/2$, $T = 1$. Columns 2-4 show errors and computational time for the new scheme. Columns 5-7 show errors and computational time for the SL

Test 2:Front propagation

For any $\gamma \in \mathbf{E}^+$, let $a_\gamma(s) : [0,1] \to \mathbb{R}^-$ be a Lipschitz function, and consider the following Hamiltonians:

$$H_\gamma(s,p) = a_\gamma(s)|p|.$$

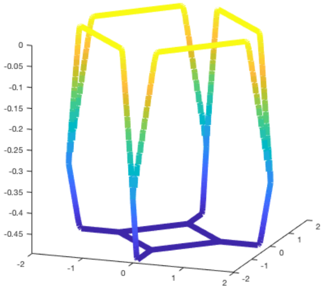
(Convergence analysis can be generalised for this case)

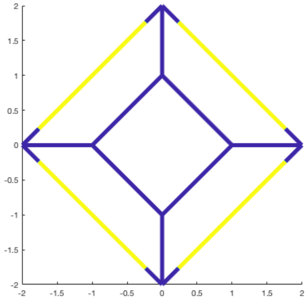


Initial condition

Test 2:Front propagation

We first set all the speeds $a_\gamma = 1$ and all flux limiters equal to 0. In this case, the flux limiter has no influence in the evolution, then an initial front given by the level set -0.2 would propagate in all the network in a time $T^* = 1 + 1.2\sqrt{2} \simeq 2.6970...$





Left: v_Δ ai time $T = 1.5$. Right: level set 0.2 at time $T = 1.5$ (blue line).

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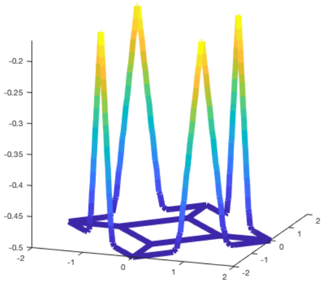

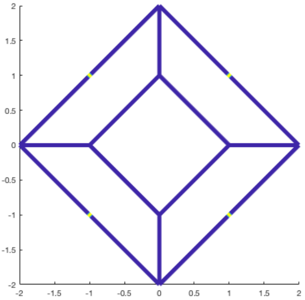

Left: v_Δ at time $T = 2.69$.Right: level set 0.2 at time $T = 2.67$ (blue line).

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