Uncertainty quantification in traffic models via intrusive method

(Joint work with M. Herty)

Elisa Iacomini

Dipartimento di Matematica e Informatica
Universita' di Ferrara

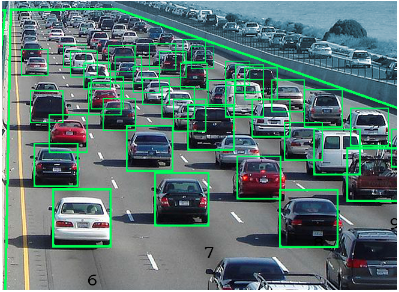
Università degli Studi di Ferrara

Framework & Motivations

Framework

Mathematical modeling of traffic flow on a **single road**, by means of:

- a microscopic (agent-based) follow-the-leader model based on ODEs
- a MACROSCOPIC (fluid-dynamic) model based on conservation laws
- a Mesoscopic (gas-kinetic) model provides a statistical description



Uncertainty

Limitations for obtaining reliable traffic forecast

- highly nonlinear dynamics
- traffic is subjected to various sources of uncertainties
  - errors in the measurements
  - estimate the reaction time of cars and drivers

Possible approaches

- non intrusive methods
  - → fixed number of samples using deterministic algorithms (i.e. Monte Carlo)
- intrusive methods
  - → reformulate the problem and solve - only once - a (big) system of deterministic equations (i.e. Stochastic Galerkin)

Stochastic Galerkin approach

- ξ uncertainty described by a random variable ω on $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$
  - ➔ we are dealing with $u(t, x, \xi) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ i.e. $\partial_t u(t, x, \xi) + \partial_x f(u(t, x, \xi)) = 0$
- the generalized polynomial chaos gPC expansion[1]:
  - ➔ we discretize the probability space Ω and the stochastic quantities are represented by infinite series expansions :

    $\phi(\xi) : \Omega \to \mathbb{R}$ orthonormal polynomials w.r.t. the inner product and $\{\phi_i(\xi)\}_{i=0}^{\infty}$ is a basis of $L^2(\Omega, \mathbb{P})$:

$$u(t, x, \xi) = \sum_{k=0}^{\infty} \hat{u}_k(t, x)\phi_k(\xi) \quad \text{where} \quad \hat{u}_k(t, x) = \int_{\Omega} u(t, x, \xi)\phi_k(\xi)\, d\mathbb{P}.$$

We can express the mean and variance of $u(t, x, \xi)$ as $\mathbb{E}[u(t, x, \xi)] = \hat{u}_0(t, x)$ and $\text{Var}[u(t, x, \xi)] = \sum_{k=1}^{\infty} \hat{u}_k^2(t, x)$.

[1] P. Pettersson, G. Iaccarino, and J. Nordström, *Polynomial chaos methods for hyperbolic partial differential equations*, Springer International Publishing, 2015

Stochastic Galerkin approach

Idea: expand the stochastic quantities in truncated series and then project

$$\mathcal{G}_K[u](t,x,\xi) = \sum_{i=0}^{K} \widehat{u}_i(t,x)\phi_i(\xi)$$

For any fixed (t,x), the expansion converges in the sense[2]

$$||\mathcal{G}_K[u](t,x,\cdot) - u(t,x,\cdot)||_2 \to 0 \quad \text{for} \quad K \to \infty.$$

Substituting the expansions in the evolution equations and applying the Galerkin projection lead to a deterministic system for the coefficients of the truncated series, due to the orthogonality of the basis functions, i.e. $\langle \sum_{i=0}^{K} \widehat{u}_i(t,x)\phi_i(\xi), \phi_j(\xi) \rangle = \widehat{u}_j(t,x)$

[2] R.H. Cameron, W.T Martin, *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*. Ann Math, 1947.

Traffic models with uncertainty

Microscopic traffic models

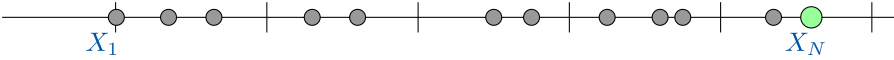
N cars on a infinite road, overtaking not possible

$x_i(t)$ position of car i at time t

$v_i(t)$ velocity of car i at time t

$a_i(t)$ acceleration of car k at time t

$$X_1 < X_2 < \ldots < X_N$$



Remark

Note that the N-th car (the leader) needs a special dynamic because has no one in front of him.

...in formulas

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First order:

$$\begin{cases} \dot{x}_i(t) = v_i(t) & i = 1, \dots N \\ v_i(t) = \begin{cases} s\left(\frac{L}{x_{i+1}(t) - x_i(t)}\right) & i = 1, \dots, N-1 \\ \bar{s}. & i = N \end{cases} \end{cases}$$

$s(\Delta x)$ is a given velocity function

Second order model:

$$\begin{cases} \dot{x}_i(t) = v_i(t) & i = 1, \dots N \\ \dot{v}_i(t) = \begin{cases} a(x_{i+1}(t), x_i(t), v_{i+1}(t), v_i(t)) & i = 1, \dots, N-1 \\ \bar{a} & i = N \end{cases} \end{cases}$$

where $a = C\frac{v_{i+1}(t) - v_i(t)}{\Delta x_i^2(t)} + \frac{A}{t_r}(s(\frac{L}{\Delta x_i(t)}) - v_i(t))$, $C, A, t_r, L > 0$

Micro model with uncertainty

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Uncertainty: Estimation of the distance between two vehicles at initial time: $x_{i+1}^0 - x_i^0 + \xi$

$\rightarrow x_i(t,\xi) \approx \sum_{k=0}^{K} \hat{x}_{i_k}(t)\phi_k(\xi)$

First order:

$$\begin{cases} \dot{x}_i(t,\xi) = v_i(t,\xi) \\ v_i(t,\xi) = s\left(\frac{L}{x_{i+1}(t,\xi)-x_i(t,\xi)}\right) \\ v_N = \bar{s}. \end{cases} \rightarrow \begin{cases} \dot{\hat{x}}_{i_k} = \hat{v}_{i_k} & i = 1,\dots N \\ \hat{v}_{i_k} = \widehat{s_{i_k}}\left(\frac{L}{\Delta x_i}\right) & i = 1,\dots,N-1 \\ \hat{v}_N = \bar{s}\ e_1 \end{cases}$$

system of $N \times (K+1)$ equations

- $\widehat{s_{i_k}} = \int_\Omega s\left(\frac{L}{x_{i+1}-x_i+\xi}\right)\Phi_k(\xi)p(\xi)d\xi$, if s is linear: $\widehat{s_{i_k}}\left(\frac{L}{\Delta x_i}\right) \approx s\left(\frac{L}{\Delta \hat{x}_{i_k}}\right)$, where $\Delta\hat{x}_{i_k} = \hat{x}_{i+1_k} - \hat{x}_{i_k}$

Micro model with uncertainty

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Second order:

$$\begin{cases} \dot{\hat{x}}_{i_k}(t) = \hat{v}_{i_k}(t) & i = 1, \dots N \\ \dot{\hat{v}}_{i_k}(t) = C\left(\mathcal{P}^{-2}(\Delta\hat{x}_{i_k})\Delta\hat{v}_{i_k}\right) + \frac{A}{t_r}\left(\widehat{s_{i_k}} - \sum_{k=0}^{\infty}\hat{v}_{i_k}(t)\right) & i = 1, \dots N-1 \\ \dot{\hat{v}}_N = \bar{a}. \end{cases}$$

system of $2N \times (K+1)$ equations

- $\mathcal{P}(\hat{u}) := \sum_{\ell=0}^{K} \hat{u}_\ell \mathcal{M}_\ell$ and $\mathcal{M}_\ell := (\langle \phi_\ell, \phi_i\phi_j \rangle)_{i,j=0}^{K}$ is a symmetric matrix of dimension $(K+1) \times (K+1)$ for any fixed $\ell \in \{0, \dots, K\}$.

Kinetic traffic flow models

$v \in [0, V_M]$ is the velocity

$g(t,x,v)$ is the mass distribution function of traffic

$Q[g]$ models the car–to–car interactions

$\varepsilon > 0$ relaxation rate towards the equilibrium

BGK type models

$$\partial_t g(t,x,v) + v\,\partial_x g(t,x,v) = \frac{1}{\varepsilon} Q[g](t,x,v), \quad g(0,x,v) = g_0(x,v)$$

- $\int_0^{V_M} g_0(x,v)\,dv = \rho_0(x)$
- $Q[g] = M_g(v;\rho) - g$ is the linear operator of BGK[a] type
- $M_g(v;\rho)$ describes the distribution at the equilibrium (Maxwellian)

[a] P. L. Bhatnagar, E. P. Gross, and M. Krook *A Model for Collision Processes in Gases I. Small Amplitude Processes in Charged and Neutral One-Component Systems*, Phys. Rev., 1954

Kinetic model with uncertainty

We are interested in the evolution of $g(t,x,w,\xi)$:

$$\partial_t g(t,x,w,\xi) + \partial_x([w - h(\rho(\xi))]g(t,x,w,\xi)) = \frac{1}{\varepsilon}(M_g(w;\rho(\xi)) - g(t,x,w,\xi))$$

$$g(0,x,v,\xi) = g_0(x,v,\xi)$$

Spectral expansion and Galerkin projection ($\sum_{i=0}^K \tilde{g}_i \phi_i(\xi)$):

$$\begin{cases} \partial_t \tilde{g}_i(t,x,w) + \partial_x \left(\left(wId - \mathcal{P}\left(h(\tilde{\rho})\right) \right) \tilde{g}(t,x,w) \right)_i = \frac{1}{\varepsilon} \left(\widetilde{M}_i\left(w;\tilde{\rho}\right) - \tilde{g}_i(t,x,w) \right) \\ \tilde{g}_i(0,x,w) = \int_\Omega g_0(t,x,w,\xi)\phi_i(\xi)p_\Xi(\xi)d\xi \end{cases}$$

where $\forall i = 0,\dots,K$:

- $\left(\mathcal{P}(h(\tilde{\rho}))\tilde{g}\right)_i = \sum_{j=0}^K \int_\Omega h\left(\sum_{\ell=0}^K \tilde{\rho}_\ell \phi_\ell(\xi)\right) \tilde{g}_j \phi_j(\xi)\phi_i(\xi)p_\Xi(\xi)d\xi,$
- $\widetilde{M}_i\left(w;\tilde{\rho}(t,x)\right) = \int_\Omega M_g\left(w;\sum_{\ell=0}^K \tilde{\rho}_\ell(t,x)\phi_\ell(\xi)\right)\phi_i(\xi)p_\Xi dwd\xi,$

Macroscopic traffic flow models

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$\rho(x,t)$ density of cars at point x and time t

$v(x,t)$ velocity of cars at point x and time t

$f(x,t) = \rho(x,t)v(x,t)$ flux of cars at point x and time t

First order model: LWR

$$\begin{cases} \partial_t \rho + \partial_x(\rho V_{eq}(\rho)) = 0, & x \in \mathbb{R},\ t > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

it is a hyperbolic conservation law where the velocity depends on the density and typically $V_{eq}(\rho) = 1 - \rho$,

Macro with uncertainty: LWR

We are interested in the evolution of $\rho(t, x, \xi)$

$$\partial_t \rho(t,x,\xi) + \partial_x(\rho(t,x,\xi)\ V_{eq}(\rho(t,x,\xi))) = 0$$
$$\rho(0,x,\xi) = \rho_0(x,\xi)$$

Spectral expansion and Galerkin projection ($\sum_{i=0}^{K} \widehat{\rho}_i \phi_i(\xi)$)

$$\partial_t \widehat{\rho} + \partial_x \left(\mathcal{P}(\widehat{\rho}(t,x))\widehat{V}_{eq}(\widehat{\rho}(t,x))\right) = \overrightarrow{0}$$
$$\widehat{\rho}(0,x) = \widehat{\rho}_0$$

with $\overrightarrow{0} = (0, \ldots, 0)^T$ vector of $K+1$ components.

Note: an arbitrary but consistent gPC expansion is required for V_{eq}, i.e. $V_{eq} = 1 - \rho$ leads to $\widehat{V}_{eq}(\widehat{\rho}(t,x)) = e_1 - \widehat{\rho}$

Macroscopic models

Second order ARZ:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & x \in \mathbb{R},\ t > 0 \\ \partial_t(v + h(\rho)) + v\partial_x(v + h(\rho)) = \frac{1}{\tau}(V_{\text{eq}}(\rho) - v), & x \in \mathbb{R},\ t > 0 \end{cases}$$

in conservative form:

$$\begin{cases} \partial_t \rho + \partial_x(z - \rho h(\rho)) = 0, & x \in \mathbb{R},\ t > 0 \\ \partial_t z + \partial_x(\frac{z^2}{\rho} - z h(\rho)) = \frac{\rho}{\tau}(V_{eq}(\rho) - v), & x \in \mathbb{R},\ t > 0 \\ v(\rho, z) = \frac{z}{\rho} - h(\rho) \end{cases}$$

- $h(\rho) : \mathbb{R}^+ \to \mathbb{R}^+$ is the hesitation function or traffic pressure law,
- $\tau > 0$ (reaction time) makes drivers tend to the equilibrium velocity. In the limit $\tau \to 0$ we recover a first order model where $v = V_{\text{eq}}$
- the system is strictly hyperbolic if $\rho > 0$.

Macro with uncertainty: ARZ

Naive idea: substitute the truncated expansions (gPC) into the random system and then use a Galerkin ansatz to project it,
i.e. $\hat{f}(\hat{\rho}(t,x)) = \langle f(\sum_k^K \hat{\rho}_k(t,x)\phi_k(\cdot)), \phi_i(\cdot)\rangle_{i=0,\dots,K}$

BUT here the Jacobian of the flux function consists of the projected entries of the deterministic Jacobian $\Longrightarrow$ not necessarily real eigenvalues and full set of eigenvectors $\Longrightarrow$ LOSS of hyperbolicity

gPC formulation for ARZ

To solve the problem:

- more assumptions on the basis functions and a change of variable, i.e. $\frac{z}{\rho}$
- derive the ARZ from the BGK approximation

gPC formulation for ARZ[3]

$$\partial_t \widehat{\rho}_i(t,x) + \partial_x \left[\widehat{z}_i(t,x) - (\mathcal{P}(\widehat{\rho}(t,x))\widehat{\rho}(t,x))_i \right] = 0$$

$$\partial_t \widehat{z}_i(t,x) + \partial_x \left[(\mathcal{P}(\widehat{z}(t,x))\mathcal{P}^{-1}(\widehat{\rho}(t,x))\widehat{z}(t,x))_i - (\mathcal{P}(\widehat{\rho}(t,x))\widehat{z}(t,x))_i \right] =$$

$$\frac{1}{\tau}\left(\Big(\mathcal{P}(V_{eq}(\widehat{\rho}(t,x)))\widehat{\rho}(t,x) + \mathcal{P}(h(\widehat{\rho}(t,x)))\widehat{\rho}(t,x) \Big)_i - \widehat{z}_i(t,x) \right)$$

$$i = 0, \dots, K$$

[3] S. Gerster, M. Herty, E. I., *Stability analysis of a hyperbolic stochastic Galerkin formulation for the Aw-Rascle-Zhang model with relaxation*, MBE, 2021.

From Micro to Macro

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Theorem (E.I.)

Let ξ be a random variable and be N cars of fixed length L. Assume that $s(\frac{L}{\Delta x}) = v(\rho)$. Then the stochastic ODEs system

$$\begin{cases} \dot{x}_i(t,\xi) = v_i(t,\xi) & i = 1, \dots N \\ v_i(t,\xi) = s\left(\frac{L}{x_{i+1}(t,\xi) - x_i(t,\xi)}\right) & i = 1, \dots, N-1 \\ v_N = \bar{s}. \end{cases}$$

converges to the stochastic LWR model

$$\partial_t \rho(t,x,\xi) + \partial_x(\rho(t,x,\xi)\ V(\rho(t,x,\xi))) = 0$$
$$\rho(0,x,\xi) = \rho_0(x,\xi)$$

for $L \to 0$ and $N \to \infty$.

Note: the same can be proven for the second order model.

From Kinetic to Macro

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Theorem (M. Herty, E.I.)

Let $\tilde{g}_i$ be a strong solution for the kinetic model for $i = 0, \ldots, K$.
Under some technical assumptions, the first and the second moment of $\tilde{g}_i$, $(\tilde{\rho}, \tilde{z})$, formally fulfill pointwise in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and for all $i = 0, \ldots, K$ the second–order traffic flow model

$$
\begin{aligned}
&\partial_t \tilde{\rho}_i(t,x) + \partial_x \left[\tilde{z}_i(t,x) - (\mathcal{P}(\tilde{\rho}(t,x))\tilde{\rho}(t,x))_i\right] = 0 \\
&\partial_t \tilde{z}_i(t,x) + \partial_x \left[(\mathcal{P}(\tilde{z}(t,x))\mathcal{P}^{-1}(\tilde{\rho}(t,x))\tilde{z}(t,x))_i - (\mathcal{P}(\tilde{\rho}(t,x))\tilde{z}(t,x))_i\right] = \\
&\quad \frac{1}{\epsilon}\left(\Big(\mathcal{P}(V_{eq}(\tilde{\rho}(t,x)))\tilde{\rho}(t,x) + \mathcal{P}(h(\tilde{\rho}(t,x)))\tilde{\rho}(t,x)\Big)_i - \tilde{z}_i(t,x)\right) \\
&\tilde{\rho}_i(0,x) = \int_W \tilde{g}_{0,i}(t,x,w)dw, \quad \tilde{z}_i(0,x) = \int_W w\, \tilde{g}_{0,i}(t,x,w)dw.
\end{aligned}
$$

Moreover, the system is hyperbolic[a] for $\tilde{\rho}_i > 0$ and and the solution is also a solution of the stochastic ARZ model.

[a] S. Gerster, M. Herty, E. I., *Stability analysis of a hyperbolic stochastic Galerkin formulation for the Aw-Rascle-Zhang model with relaxation*, MBE, 2021.

Diffusion coefficient

Starting from

$$\partial_t g(t,x,w,\xi) + \partial_x \Big[(w - h(\rho(t,x,\xi)))g(t,x,w,\xi) \Big] = \frac{1}{\varepsilon}\Big(M_g(w;\rho) - g(t,x,w,\xi) \Big)$$

- assume $\varepsilon > 0$: small but positive.
- perform a **first-order** Chapman Enskog approximation
  - → $g(t,x,w,\xi) = M_g(w;\rho(t,x,\xi)) + \varepsilon g_1(t,x,w,\xi)$
- obtain an **advection-diffusion** equation [4].

$$\partial_t \rho + \partial_x \left(\rho V_{eq}(\rho) \right) = \epsilon \partial_x \left(\mu(\rho) \partial_x \rho \right), \quad \rho = \rho(t,x,\xi),$$

$$\mu(\rho) = \Big(-\partial_\rho Q_{eq}(\rho)^2 - \partial_\rho h(\rho) \partial_\rho Q_{eq}(\rho)\rho + Q_{eq}(\rho)\partial_\rho h(\rho) \Big) + \int_V v^2 \partial_\rho M_f(v,\rho) dv$$

Tool for studying possible instabilities

$$\mathbb{P}_{t,x}(\mu \le 0) := \int_\Omega H(-\mu(\rho(t,x,\xi)) p_\Xi(\xi) d\xi.$$

[4] M. Herty, G. Puppo, S. Roncoroni, G. Visconti, *The BGK approximation of kinetic models for traffic*, Kinetic & Related Models, 2020.

Numerics

Numerical settings

- $\xi \sim \mathcal{U}(0,1)$,
- Basis choice: Haar basis

$$\psi(\xi) := \begin{cases} 1 & \text{if } 0 \leq \xi < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leq \xi < 1, \\ 0 & \text{else.} \end{cases} \quad \text{and} \quad \psi_{j,k}(\xi) := 2^{\frac{j}{2}} \psi\big(2^j \xi - k\big)$$

  Using a lexicographical order we identify the gPC basis $\phi_0 = 1$, $\phi_1 = \psi$, $\phi_2 = \psi_{1,0}$, $\phi_3 = \psi_{1,1}, \ldots$

- $\Delta x = 2 \cdot 10^{-2}$ on the space interval $[0,2]$,
- $T_f = 1$ and Δt fulfills the CFL condition,
- $h(\rho) = \rho$,
- $V_{\text{eq}}(\rho) = 1 - \rho$.

Fundamental Diagram

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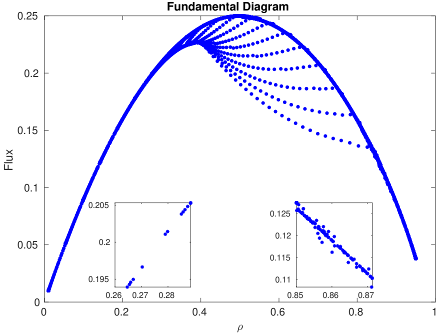
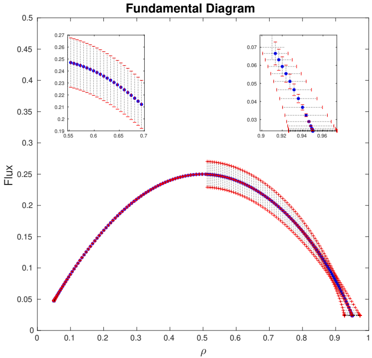
Mean



Mean & Variance



Numerical settings

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Initial data

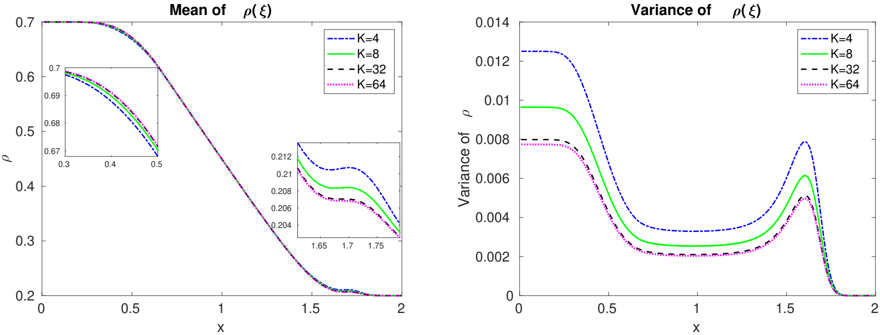
Rarefaction wave:

$$\rho(x,0,\xi) = \begin{cases} 0.55 + 0.3\xi & \text{for} \quad x < 1, \\ 0.3 & \text{for} \quad x > 1, \end{cases} \qquad v(x,0,\xi) = \begin{cases} 0.2 & \text{for} \quad x < 1, \\ 0.7 & \text{for} \quad x > 1. \end{cases}$$

Clever idea

We compute offline in a precomputation step the entries of the matrices $\mathcal{P}(\cdot)$ and the tensor $\mathcal{M} \implies$ **not** computationally expensive.

Numerical convergence in K



Application: detect high risk regions

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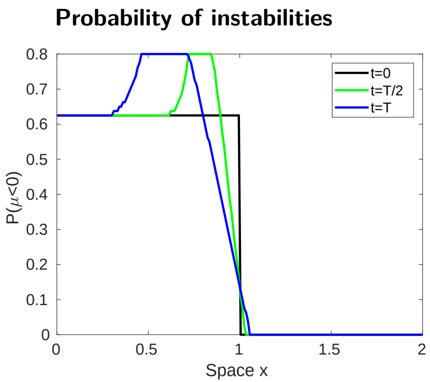
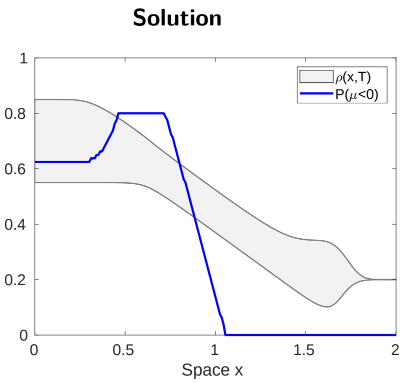
$$\mathbb{P}_{t,x}(\mu \leq 0) := \int_{\Omega} H(-\mu(\rho(t,x,\xi))p_{\Xi}(\xi)d\xi.$$





Conclusion and future perspectives

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Recap

- Uncertainty is introduced in traffic flow models to improve traffic forecast.
- Micro, kinetic and macroscopic scales are investigated and the convergence to the latter one is shown. Moreover the obtained formulation preserves hyperbolicity.
- The stability analysis is performed and the diffusion coefficient is studied.
- Numerical simulations illustrate the theoretical results.

What's next

- Use real data to estimate ξ.
- Study the uncertainty in the non-local case.
- study the uncertainty via "efficient" data-friendly non-intrusive methods.

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Thank you for your kind attention!

Elisa Iacomini – elisa.iacomini@unife.it

Dipartimento di Matematica e Informatica
Universita' di Ferrara
Via Machiavelli 30
44121 Ferrara

Numerical scheme

Idea: employ the local Lax Friedrichs scheme to solve SG-ARZ combined with an IMEX scheme[1] in the inhomogeneous case. The source term is treated implicitly, due to the stiffness while the convective term is treated explicitly for $l = 0, \ldots, K,\ j = 0, \ldots, N$:

$$\text{Expl. update} \begin{cases} \bar{\rho}_{j,l}^{n+1} = \bar{\rho}_{j,l}^{(1)} - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}(\bar{\rho}^{(1)}, \bar{z}^{(1)}) - F_{j-\frac{1}{2}}(\bar{\rho}^{(1)}, \bar{z}^{(1)})\right) \\ \bar{z}_{j,l}^{n+1} = \bar{z}_{j,l}^{(1)} - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}(\bar{\rho}^{(1)}, \bar{z}^{(1)}) - F_{j-\frac{1}{2}}(\bar{\rho}^{(1)}, \bar{z}^{(1)})\right) \end{cases}$$

$$\text{Implicit step} \begin{cases} \bar{\rho}_{j,l}^{(1)} = \bar{\rho}_{j,l}^{n} & l = 0, \ldots, K,\ j = 0, \ldots, N \\ \bar{z}_{j,l}^{(1)} = \frac{\tau}{\tau+\Delta t}\bar{z}_{j,l}^{n} + \frac{\Delta t}{\tau+\Delta t}\left(\mathcal{P}(\bar{\rho}^{n})\widehat{V_{eq}}^{n} + \mathcal{P}(\bar{\rho}^{n})\bar{\rho}^{n}\right) \end{cases}$$

where $F_{j\pm\frac{1}{2}}$ is the numerical flux of the local Lax Friedrichs scheme.

[1] L. Pareschi and G. Russo, *Implicit–explicit runge–kutta schemes and applications to hyperbolic systems with relaxation*, Journal of Scientific computing, 2005

Hyperbolic formulation

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Assumptions on basis functions

A1) The precomputed matrices $\mathcal{M}_\ell$ and $\mathcal{M}_k$ commute for $\ell, k = 0, \dots, K$.

A2) There is an eigenvalue decomposition $\mathcal{P}(\hat{u}) = V\mathcal{D}(\hat{u})V^\mathsf{T}$ with constant eigenvectors.

A3) The matrices $\mathcal{P}(\hat{u})$ and $\mathcal{P}(\hat{y})$ commute for all $\hat{u}, \hat{y} \in \mathbb{R}^{K+1}$.

SG hyperbolic preserving formulation

Moreover, assuming $h(\rho) = \rho^\gamma, \gamma = \{1, 2\}$, so $\hat{h}(\hat{\rho}) = \mathcal{P}^{\gamma-1}(\hat{\rho})\hat{\rho}$, and its Jacobian of the form $\widehat{h'}(\hat{\rho}) = V\mathcal{D}_{\hat{h}'}(\hat{\rho})V^\mathsf{T}$, we get

$$\begin{cases} \partial_t \hat{\rho} + \partial_x \left(\hat{z} - \mathcal{P}(\hat{\rho})\hat{h}(\hat{\rho}) \right) = \vec{0} \\ \partial_t \hat{z} + \partial_x \left(\mathcal{P}(\hat{z})\mathcal{P}^{-1}(\hat{\rho})\hat{z} - \mathcal{P}(\hat{z})\hat{h}(\hat{\rho}) \right) = \vec{0}. \end{cases} \tag{2}$$

Main result

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Theorem 1

Let a gPC expansion with the properties (A1) – (A3), a stochastic Galerkin formulation of a hesitation function $\hat{h}(\hat{\rho})$ and a Galerkin formulation of an equilibrium velocity $\widehat{V_{\text{eq}}}(\hat{\rho})$ be given. Assume further a Jacobian of the hesitation function

$$\widehat{h'}(\hat{\rho}) = \mathsf{D}_{\hat{\rho}}\hat{h}(\hat{\rho}) = V\mathcal{D}_{h'}(\hat{\rho})V^T$$

with constant eigenvectors.
Then, for **smooth** solutions (**??**) and (**??**) are equivalent and strongly hyperbolic. The characteristic speeds are

$$\widehat{\lambda_1}(\hat{\rho},\hat{z}) = \mathcal{D}\big(\hat{v}(\hat{\rho},\hat{z})\big) - \mathcal{D}_{h'}(\hat{\rho})\mathcal{D}(\hat{\rho}) \quad \text{and} \quad \widehat{\lambda_2}(\hat{\rho},\hat{z}) = \mathcal{D}\big(\hat{v}(\hat{\rho},\hat{z})\big)$$

for $\hat{v}(\hat{\rho},\hat{z}) = \mathcal{P}^{-1}(\hat{\rho})\hat{z} - \hat{h}(\hat{\rho})$, where $\mathcal{D}(\hat{v})$ denote the eigenvalues of the matrix $\mathcal{P}(\hat{v})$.

Stability analysis

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Theorem 2

Under the same assumptions of the previous Theorem, the first-order correction to the local equilibrium approximation reads

$$\partial_t \hat{\rho} + \partial_x \widehat{\mathbf{f}_{\text{eq}}}(\hat{\rho}) = \tau \partial_x \big(\hat{\mu}(\hat{\rho}) \partial_x \hat{\rho} \big), \qquad \widehat{\mathbf{f}_{\text{eq}}}(\hat{\rho}) = \hat{\rho} * \widehat{V_{\text{eq}}}(\hat{\rho})$$

$$\hat{\mu}(\hat{\rho}) = V \left[\mathcal{D}(\hat{\rho})^2 \mathcal{D}_{V_{\text{eq}}}(\hat{\rho}) \Big(\mathcal{D}_{V_{\text{eq}}}(\hat{\rho}) + \mathcal{D}_h(\hat{\rho}) \Big) \right] V^T.$$

Furthermore, it is dissipative if and only if the sub-characteristic condition

$$\widehat{\lambda_1}(\hat{\rho}, \hat{z}) \leq \widehat{\mathbf{f}'_{\text{eq}}}(\rho) \leq \widehat{\lambda_2}(\hat{\rho}, \hat{z})$$

holds on $\hat{z} = \hat{\rho} * \Big(\widehat{V_{\text{eq}}}(\hat{\rho}) + \hat{h}(\hat{\rho}) \Big)$ with $\mathcal{D}_{V_{\text{eq}}}(\hat{\rho}) < \vec{0}$.